Modeling mechanical systems

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Introduction

Previously we've used a relatively ad-hoc approach to come up with mechanical models and their electrical equivalents. While that approach works for simple systems, it quickly becomes error-prone. In this article I will describe a more systematic approach that can be used to analyze mechanical systems alone, or combined electro-mechanical systems together.

In the end, the circuits derived with this method can also be used in for example circuit simulators to analyze and simulate behavior, but the results are easy to relate back to the mechanical world.

Another important benefit is that this model preserves enough of the mechanical details that mechanical systems can easily be cascaded as we'll see in the examples.

Finally, the description here can easily be adopted for other electro-mechanical devices than motors: speakers, microphones, even MEMS devices, like accelerometers or gyroscopes.

The method I'll introduce here will deal with 1DOF, rotational mechanical systems. It is easy to develop a similar model for 1DOF translational systems as well.

Mechanical Schematics

We start the discussion with a way of describing (1DOF, rotational) mechanical systems similar to the way we do it with electrical circuits: by drawing a schematic representation of it. The goal is to develop a method that represent the topology (connection) and the various components that make up a mechanical system, but hiding all the details of actual construction, size, implementation detail etc. In order to do so, we'll have to introduce a way to extract the components and their connections. I will roughly follow the description used here:

http://lpsa.swarthmore.edu/Systems/MechRotating/RotMechSysElem.html

but as usual, I will make some changes to make my points more clear.

Just as with electrical systems, our mechanical schematics will be special graphs: comprised of nodes and connections (or vertices). We will put special notations on these elements and assign special meaning to them to describe things.

Nodes

In electrical design, we choose to represent points that share the same potential with nodes (occasionally we extend nodes with lines to make the schematic more readable, but that's irrelevant here). We also choose to represent current flows by placing connections between these nodes and draw symbols on these connections that describe the relationship between the current and the potential-difference (voltage) between the two nodes they connect. Please understand that while this representation feels natural, it is actually arbitrary and we could just as well as go the other way around (see dual graphs for an idea on how) and get a completely useable model as well.

In our mechanical world, we also have two measurable properties to deal with: torque and rotational velocity (speed), In systems with only 1DOF, both of these quantities are scalars, just as voltage and current are in electrical systems. The representation that I'll use in this explanation will be such that I use nodes to represent points that share the same speed – shafts for the most cases. Wherever torque can transfer from one shaft to another, I will draw connections between the corresponding nodes, and I'll place symbols on these connections to describe the relationship between the torque and the speed-difference between the two nodes these vertices connect.

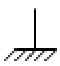
While this representation is also arbitrary, it also feels at least somewhat natural: the nodes abstract shafts, that connect components that rotate together at the same rate.

Ground

There's one special node in both electrical and mechanical systems that are worth mentioning: the ground point. This is the node with 0 potential in an electrical system and serves as a reference to measure all other potentials in the system. It reflects the fact that the electrical field – at least the ones we care about in electrical schematics – is conservative.

In the mechanical world, there's also a special node like that, a node which has 0 speed. It also serves as a reference to measure all other speeds in the system and expresses the fact that we operate in an inertial reference frame.

In the following I'll denote this special note with this symbol:



Just as with electronics, if multiple ground symbols are in a schematic, the interpretation is that all such places are in fact connected to the same, unique ground node.

Mechanical component characteristics

Mechanical components in our model have two external phenomena effecting them: speed (rotational velocity) and torque. All of these elements have some internal characteristics, describing their response to these external effects. These characteristic equations are different for each type of component, but in general establish a relationship between torque and speed on the mechanical component. Let’s see a few of these components!

Damping

Viscous friction, or damping is probably the simplest of all mechanical component to model. This component acts as a loss and it introduces a torque on the shaft that is proportional to the speed of the rotation. The torque introduced by is in the opposite direction to the rotation of the shaft as damping drains energy out of the system. In the following however we’re going to look at the components from the systems’ perspective. In that view, we need to ‘feed’ some torque into the component to compensate for its internal torque. Because of that, I leave the negative sign out:

T = d*s, where d is a constant, describing the lossyness of the component.

I will use this symbol to represent damping components:

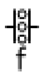
As you can see, this component has two connections, so it supposed to connect two shafts. It is there to model the fact that all types of friction, including damping is a relative phenomena: friction acts between two bodies which move at different speeds relative to each other.

Kinetic Friction

Kinetic friction or friction in short introduces a constant torque that counter-acts the speed of the shaft. As we will use it in the discussion below, friction is only slightly non-linear:

T = sign(s) * f, where f is a constant describing the loss on the component

The sign of the torque is again viewed from the systems’ perspective. The symbol I’ll use for this type of friction is the following:



The reason I say that friction is only slightly non-linear is that if we assume that the shaft is rotating in one (let’s say positive) direction only, the torque on the friction is going to be constant.

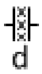
Static friction

Static friction is an even more non-linear element than its kinetic brethren. The best way of describing it in our representation is as follows: the speed-difference between the two ends of it is 0 as long as the torque is under a certain (absolute) limit – it acts as a 'short circuit'. When the torque reaches that certain limit, the component 'disappears' completely and doesn't enforce any relationship between the speeds of its two ends. At that point it doesn't 'consume' any torque either, so it becomes an 'open circuit'. It stays in that 'open' state as long as the speed-difference reaches 0 again, at which point it 'locks' and returns to the 'short circuit' behavior. The equations describing this behavior are the following:

s = 0 if abs(T) 0

Please note, that this type of friction is not only non-linear but a hysteretic element as well. It has two states and it switches between them not only depending on its current conditions but on past history as well. This behavior makes it particularly hard to work with and in many models assumptions are made as to which state static friction is in, and is assumed to never change state.

The symbol for static friction is:

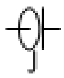


Mass

A rotating mass resist **changes in speed** or acceleration, so the torque it introduces will be proportional to the derivative of the speed. The characteristic equation is the following:

T = J * (ds/dt), where J is the inertia of the mass

The symbolic representation will be the following:



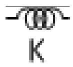
Now, here's one curiosity: why does mass have two connections? The answer is that while mass or inertia isn't relative, classical mechanical systems are. They can only be interpreted in a so-called [inertial reference frame](). Movement of masses (consequently their speed or acceleration) is always relative to the chosen reference frame. We denote this reference frame with the ground node so, one of the connection of the mass symbol is always connected to this ground node.

Springs

Let's also take a look at the other reactive element in mechanical designs: the spring. Springs exert a force (or torque in a rotational system) that's proportional to **displacement** or the integral of the speed as opposed to the speed itself. Here we talk about displacement or speed in relative terms: the displacement of one end of the sprig relative to the other. Another way of stating this is that the change of the torque of a sprint is proportional to the speed:

dT/dt = K * s where K is the spring constant

The symbol for it will be:



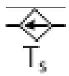
Torque source

Now that we're done with the passive elements, lets introduce some active ones as well. These would be components that somehow (form an external source) pump mechanical energy into the system. The first type we'll introduce is a torque source. This puts a constant torque into a system, independent of its speed:

$T = T_s$

It is represented by this symbol:

If I want to emphasize that the torque introduced by the component is a function of something, I'll use the controlled torque source symbol:



Speed source

The final component that I'll introduce for now is the speed source. This, as it's name shows introduces a constant (relative) speed into the system independent of how much torque it takes to do so:

$s = s_s$

It's symbols for simple and controlled cases are:

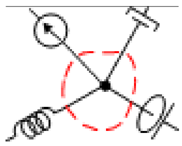
and

Now it’s important to understand that all of these components, but especially the sources are highly idealized. They are good as building blocks and for theoretical discussions but they are not much more than rough first approximations of actual physical components.

Circuit laws

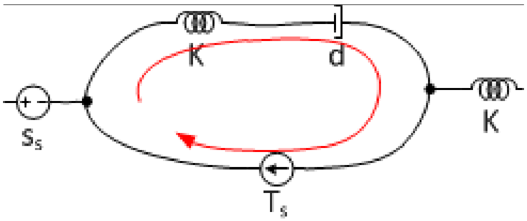
The last thing we will have to establish is some laws that govern our schematics. In electronics, these are the Kirchhoff's circuit laws and we’ll need the equivalents as well. The two similar laws for us are the following:

1: For any node in the system the sum of the torque through all connecting elements to that node is 0.



This law essentially a re-statement of Newton’s third law in a rotational system.

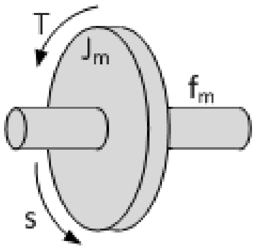
2: For any loop in the system the sum of the speed-differences on all the elements making that loop is 0.



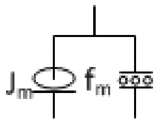
This second law is a consequence of us working in an inertial reference frame.

An Example: Revisiting the DC motor

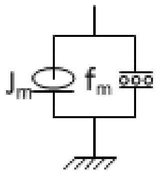
Well, that was dry, but finally we’re at the point to put all that in use! Let’s take another look at our old DC motor. The mechanical side is usually modeled by a rotating disc and some friction:



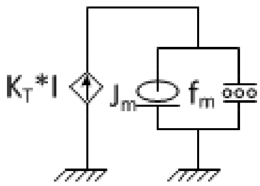
Let's use our newly acquired knowledge and draw a mechanical schematic representation of it! We have two mechanical components to consider: the rotating mass and the friction. These two components are connected to the same shaft, so the one end of the two symbols are connected together:



But what happens with the other end? Well, for the mass it's connected to ground as all masses in a mechanical system are, but what about the friction? Remember what we model with this fiction is the mechanical losses, mostly in the bearings. The other end of these bearings is tied to the casing of the motor, so the other 'shaft' that this fiction component is connected to is the housing of the motor. If we assume that the housing of the motor is stationary, it doesn't rotate relative to ground. If that's the case, we can model the friction as if is connected to ground as well:



There are two things missing: the source of the mechanical energy and the mechanical load. If we model the motor alone, the load is simple: we don't have it. The source isn't that complex either: it is the torque converted by the motor. This torque is acting between the shaft and the housing of the motor, and as we've discussed before, the we assume that the motor housing is stationary:



This model can be used (using the circuit laws above) to reason about all sorts of things. For example:

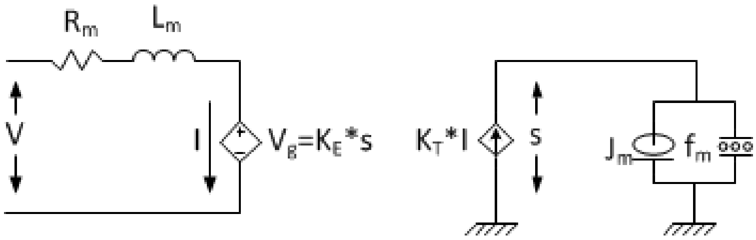
There are two nodes in the system (the motor shaft and ground). Consequently all three mechanical components are connected in parallel. This means that all share the same speed, but the torque on them is different. Of course the torque on the source is constant (K_T*I), and in our model, friction also has a constant torque at least as long as the direction of the motion doesn't change. So, some of the torque converted by the motor is used up in the friction, the rest will appear on the mass. If you look at the characteristic equation for the mass however:

$T = J_m * (ds/dt)$,

You'll see that constant torque will create a constant change (acceleration if it's positive) in speed. Neither the friction nor the torque source put any other constraints on the speed, so the speed will in deed start changing. The torque on the torque source is proportional to the current in the motor windings, so we get the following:

If we put a constant current through the motor, we should see a constant acceleration on the shaft.

Well, that can't be accurate, can it? Have you ever seen a motor keep accelerating forever, even if we don't have any load in its shaft? No, of course not, and we'll see in a minute why! The first thing that should be easy to see is that we don't usually put a constant current through the motor, but a constant voltage. The other thing should be mentioned however is the whole electrical side of the motor: the inductor, the internal resistance, and the generator voltage. A motor model isn't complete without them:



At first it looks quite a bit complicated, but in fact it really isn't: it only makes the electrical/mechanical conversion in the motor explicit. It shows both the controlled voltage source (which is the conversion in the mechanical –> electrical direction) and the controlled torque source (which is the conversion in the electrical –> mechanical direction).

Now, if we look at the complete picture, and imagine the idle state, where no voltage is applied and the shaft is not rotating, than apply a constant voltage (V) to the motor, the following will happen:

1. The motor current starts increasing linearly due to the inductance. For now, you have to believe me that the time constants on the mechanical side are much higher than on the electrical side, so V_g remains close to 0 (the shaft doesn't start rotating fast enough)

long enough that the current reaches its maximum, where all the input voltage is dropped on the internal resistance (R_m). During this time, the current raises exponentially in the circuit.
2. This exponential rise in current will translate into an exponential rise in torque on the torque source. Now our friction model is not accurate enough to represent the case where the torque sourced is lower than the torque consumed by the friction, yet the shaft is not moving – that would be modeled by a static friction component – but for now, let's ignore that little detail, and assume that the torque source pretty soon reaches torque values that are higher than the torque consumed by friction (f_m). This torque will start accelerating the mass of the motor, but – and again you'll have to take my word for it here – it reaches its maximum before it can speed up the mass by any significant amount.
3. At this point, the maximum current is flowing through the motor and the mass slowly starts accelerating. The electrical transient is over, now the combined electro-mechanical one follows. As we've discussed before a constant current generates a constant acceleration on the mass, and that is exactly what's happening in the beginning. However as soon as the shaft starts rotating, it will introduce a generator voltage ($V_g = K_m$*s), that starts counter-acting the source voltage (V).
4. As the speed increases, so does the generator voltage (V_g), and as V_g rises, less and less voltage drops on the resistor. As a consequence, the current through the motor will start falling, and as it does, so does the torque on the mechanical side. Less torque generates less acceleration on the mass, which means less change in speed, which translates slower rise in V_g on the electrical side.
5. The result of this interaction is another exponential transient, where the mass of the motor and the internal resistance of the winding determine the time-constant.
6. When this transient finishes, neither speed, torque, voltage or current changes in the device any more. This means that the torque on mass must be 0, but that implies that the torque generated by the torque source must be equal to the torque consumed by the friction. It also means that the generator voltage must be high enough that the voltage drop on R_m (which determines the current through the motor) is just enough to generate that torque.

The takeaways from this long-winding example are the following:

- Dealing with electro-mechanical systems, like a motor involves dealing with both electrical and mechanical aspects of it
- Some time-constants in the system are defined by purely mechanical or electrical components, however some could be determined by an interaction between the two
- Conversion equations and controlled-sources are important tools to describe relationships between electrical and mechanical quantities
- Combined systems are a pain in the rear to deal with

So is there a better way? We'll explore that in the next chapter

Electrical Equivalent Circuits

By now you probably have seen quite a few parallels between the ways we describe mechanical and electrical systems. Let's imagine the following: we want to construct an electrical circuit that is a model of the mechanical circuit we would like to analyze! We want to do it so, that all the measurements and calculations we do in the electrical domain are immediately applicable to the mechanical equivalent.

The first order of business: how do we map our measureable quantities – torques and speeds – to voltages and currents? The natural mapping comes from the circuit laws. Once again, they are:

1: For any node in the system the sum of the torque through all connecting elements to that node is 0.

2: For any loop in the system the sum of the speed-differences on all the elements making that loop is 0.

The first law is similar to Kirchhoff's current law: at any node in an electrical circuit, the sum of currents flowing into and out of that node is 0.

The second law is the equivalent of Kirchhoff's voltage law: the directed sum of the potential differences (voltage) around any closed circuit is zero.

So, the natural mapping of these parameters is to represent torque with current and speed with voltage.

We have to come up with ratios between torque and current and speed and voltage. Since we're building only electrical models that make it easy to us to make calculations and predictions about mechanical systems, this factor can be anything convenient. We can say that 1A of current represents 1Nm of torque, but we could just as well as say that 1mA of current represents 1 oz" of torque, it doesn't matter. Same goes for voltages and speeds. The only thing that matters is that we choose one conversion and we stick to it.

In the following I will use the mapping that 1A current represents 1Nm of torque and 1V voltage represents 1rad/s speed, but as I said before, this is arbitrary.

Components

The next order of business is to come up with electrical components that can be substituted for our mechanical ones. For this, we'll look at the characteristic equations.

Drag

Drag has a very simple one:

T = d*s

We know now, that will represent torque with current and speed with voltage, so our electrical equivalent should have a characteristic equation of the following form:

I = d*V

This is a **resistor**, with a resistance of ‘d’.

Kinetic Friction

Friction is a little more complex, being a non-linear component. The original characteristic equation was:

T = sign(s) * f

Doing the replacement we get:

I = sign(V) * f

This can be modeled with a **controlled current source**, that changes direction depending on the voltage applied to it. And just as in the mechanical version, this model can be simplified to a constant current source if you assume that voltage never changes polarity.

Static Friction

Static friction is even more troublesome. The characteristic equation is not only non-linear but hysteretic as well:

s = 0 if abs(T) 0

After replacement we get:

V = 0 if abs(I) 0

This is either a **short or an open circuit** depending on the state of the element.

Mass

Masses equation was the following:

T = J * (ds/dt)

After replacement we get:

I = J * (dV/dt)

This is what a **capacitor** does, if it has a capacitance of ‘J’.

Spring

It is probably not a big surprise after all this that springs can be represented by **inductors** with an inductance of ‘1/K’, bacuse

dT/dt = K * s

becomes

dI/dt = K * V

Torque source

Sources are almost trivial to convert: since torque becomes current, torque sources become **current sources**.

Speed source

Similarly, speed sources are converted into **voltage sources**.

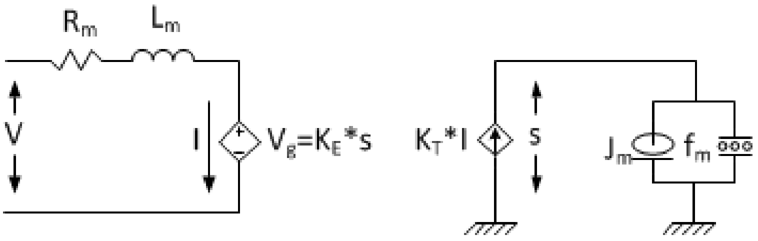
Summary

To summarize, during the conversion to an electrical circuit, the following replacements are made:

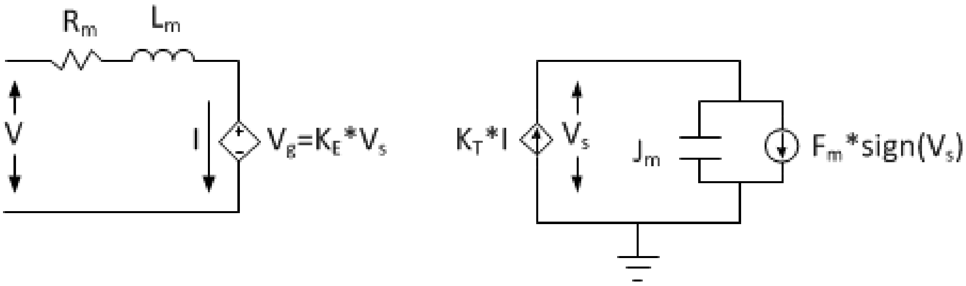
Original	**Converted**
Torque	Current
Speed	Voltage
Drag	Resistor (R = d)
Kinetic Friction	Non-linear, controlled current-source
Static Friction	Hysteretic open or short circuit
Mass	Capacitor (C = J)
Spring	Inductor (L = 1/K)
Torque Source	Current Source
Speed Source	Voltage Source

Another look at the DC motor

As our first example, let’s continue where we left off with modeling our old DC motor:



Now, let's use our newly acquired knowledge and replace the left-side with their corresponding electrical symbols:

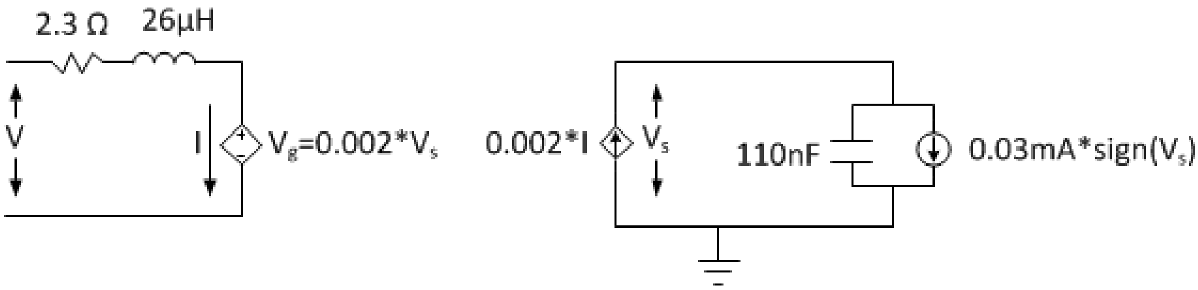


As you can see, I've replaced the speed with voltage (V_s), all the components with their electrical equivalent and the mechanical ground with the electrical one. This latest step is not really relevant. In fact, when you replace the mechanical ground, you can replace it with any constant voltage node. Just make sure you keep some absolute and constant reference around for the mechanical portion.

You now have a completely electrical circuit, that you can reason about using all the familiar techniques of circuit analysis. I'll do just that, but first, let's put some numbers in place! For this example, I'll take the [1024-003s motor from Faulhaber] because it has a fairly detailed datasheet. From that:

R_m = 2.3 ?
L_m = 26 µH
K_E = 0.215 mV/rpm = 0.002 V/(rad/s)
K_T = 2.05 mNm/A = 0.002 Nm/A
J_m = 0.12 gcm^2 = 0.11 * 10^{-6} Nm^2
F_m = 0.03 mNm = 0.03 * 10^{-3} Nm

If we represent 1Nm torque with 1A of current and 1rad/s of speed with 1V of voltage, we get the following electrical equivalent:



This is a circuit that you can put in a circuit simulator (if you do, be careful about the polarities of the sources!) and plot the frequency response for example (here I plot the generator voltage):

You see the expected low-pass characteristics, with two corner-frequencies. One at around 3Hz, the other at around 20kHz. This is very similar to what we've seen with our [simple motor modeling]() exercise. Of course the values are different as we're dealing here with a different motor.

The interesting thing in this model however, is that it preserves the mechanical quantities as well. For example, in the following I'll put a 3V pulse for 1s on the motor and plot the voltage across the capacitor:

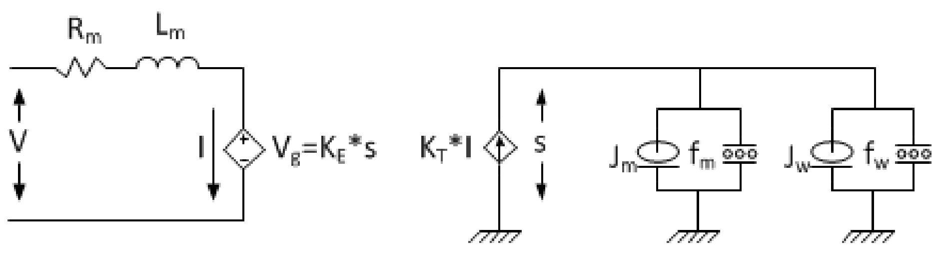
Now, we know that we modeled speed with voltage, so we're in fact looking at a speed-over-time graph here. It shows that when the voltage is turned on, it takes some time – around 250ms – for the motor to reach the target speed. When the transient is finally over, we measure 1.48kV on the capacitor. (Don't be afraid of the big numbers, these are not real voltages, just equivalent numbers.) We've represented 1rad/s speed with 1V, so it must be 1480 rad/s speed. Converting it to rpm, we get 14132. This is close to the 13800 rpm value, specified in the datasheet. The difference is probably in the way we model our losses versus how they actually are in the motor. 3% accuracy is pretty good for such a simple model.

We can also plot the current through the capacitor:

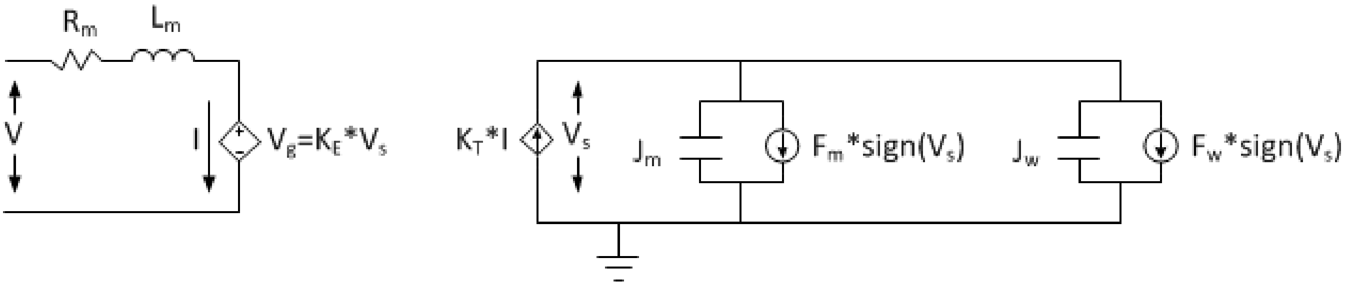
Here you see that the torque decreases to 0 (in fact the only torque converted on the motor is used up by friction) after the initial transient. The peak current is at 2.5mA. This corresponds to 2.5mNm torque on the rotating disc. There's a small amount of torque that's consumed by the fiction so the motor needs to output a little more than that.

While this example is not terribly enlightening unless our plan is to use the motor without any load, but shows that we can directly obtain mechanical properties from the equivalent electrical circuit.

The last thing I would like to point out is this: since the we've preserved the topology of the mechanical side of the system, we still have a representation for the output shaft of the motor. So, if we want to now tie a load to the motor (let's say a large wheel with some friction), we can do that. All we need to do is extend our mechanical model with the new components:



Here J_w and f_w are the inertia and the friction of the wheel respectively. Now, we can re-draw our electrical equivalent:



We can use this new model to reason about the changed circuit. For example, we see that J_m and J_w are in parallel, so their capacitance will add together. This means that the lower corner-frequency in our frequency response plot (which is determined by the capacitance and R_m) will move to a lower frequency. In other words, if you add a wheel to a motor, it will become more sluggish to respond to voltage changes, which is not that surprising. But now we also see that we can't move that corner-frequency higher – at least not with passive circuitry. For that we would need to either reduce the capacitance – which has a minimum set by the inertia of the rotor – or the series resistance – which also has a minimum set by the internal resistance of the coils. Our only chance to make our system respond faster is to use active elements, in other words, introducing a control mechanism.

Final notes

If you've gotten this far, let me share with you a couple of final remarks:

The method I've introduced here is sometimes called the 'mobility analog' and you'll find more examples on the internet by searching for that term. There's another way of modeling mechanical systems, called the 'impedance analog' that relates force (torque) to voltage and velocity (speed) to current. This might seem physically more accurate at first as voltage is in fact related to force exerted on charged particles in an electric field and current can be thought of as speed of charged particles through a certain surface. In fact that was the way we've taught to do this in my university. However as I said in the beginning, the choice is actually arbitrary and the two ways are equivalent in terms of the results you can obtain from them. Dealing with the impedance analog however is more troublesome at least in my experience as it requires to draw the dual graph of the mechanical system to arrive to the electrical equivalent.

I'm also planning another article where I'll give more examples of how to use the technique I've introduced here to model more complex electro-mechanical systems and gain important insights into their behavior.